

COHOMOLOGICAL APPROACH TO ASYMPTOTIC DIMENSION

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ABSTRACT. We introduce the notion of asymptotic cohomology based on the bounded cohomology and define cohomological asymptotic dimension $\text{asdim}_{\mathbf{Z}} X$ of metric spaces. We show that it agrees with the asymptotic dimension $\text{asdim } X$ when the later is finite. Then we use this fact to construct an example of a metric space X of bounded geometry with finite asymptotic dimension for which $\text{asdim}(X \times \mathbf{R}) = \text{asdim } X$. In particular, it follows for this example that the coarse asymptotic dimension defined by means of Roe's coarse cohomology is strictly less than its asymptotic dimension.

§0 INTRODUCTION

Gromov proposed to study discrete groups as large scale geometric objects. He introduced several asymptotic invariants of finitely generated groups [Gr1]. Among them there is the notion of asymptotic dimension which proved to be useful for the Novikov-type conjectures [Yu],[Ba],[CG],[Dr2],[DFW]. The asymptotic dimension differs from any other known dimension invariant of discrete groups. Thus, it differs from the geometric dimension $gd(\Gamma)$ [Br] for every finite group Γ since the geometric dimension is infinite for groups with torsions and the asymptotic dimension of finite groups is zero. If one takes into account only torsion free groups then the distinction between these two dimensions is not obvious. For almost all known torsion free groups where both invariants are computed there is the equality $\text{asdim } \Gamma = gd(\Gamma)$. An exception is Gromov's example of a group Γ_0 with finite classifying space $B\Gamma_0$ that contains an expander (in some weakly coarse sense) [Gr2]. This group does not admit a coarse embedding into a Hilbert space and hence [HR], $\text{asdim } \Gamma_0 = \infty$ whereas $gd(\Gamma_0) < \infty$.

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For discrete groups it is known that modulo some potential discrepancy in the case of geometric dimension three (Eilenberg-Ganea problem) there is the equality $gd(\Gamma) = cd(\Gamma)$ where $cd(\Gamma)$ is the cohomological dimension of Γ with integral coefficients [Br]. This equality gives an effective way for computation of $gd(\Gamma)$. A similar situation happens to be in the classical dimension theory of locally compact metric spaces. One of the basic facts there is the Alexandroff Theorem which states that the covering dimension agrees with the integral cohomological dimension: $\dim X = \dim_{\mathbf{Z}} X$, provided $\dim X < \infty$. The cohomological approach in the classical dimension theory very often allows one to compute the covering dimension or to reduce the computation to a calculation of cohomological dimensions of X with respect to the coefficient groups \mathbf{Q} , $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$, and $\mathbf{Z}_{p^\infty} = \lim_{\rightarrow} \mathbf{Z}_{p^k}$ for prime p . Also it gives an exact formula for dimension of the product (Bockstein's formula [Dr3]). In particular the cohomological approach to dimension allows to prove Morita's theorem which states that $\dim(X \times \mathbf{R}) = \dim X + 1$ for general topological spaces X .

In the light of all this it would be useful to have a cohomological approach to the asymptotic dimension. The right approach would give answer to many naive still open questions in asymptotic dimension theory, like, *Does the asymptotic Morita theorem hold true: $\text{asdim}(\Gamma \times \mathbf{Z}) = \text{asdim } \Gamma + 1$ for finitely generated groups Γ ?*

Perhaps the first attempt to define asymptotic dimension cohomologically was made in [Dr1] by means of Roe's coarse cohomology theory HX^* . We redefine the corresponding dimension in this paper (§4) and call it the coarse cohomological dimension $X\text{-dim}_G$. It turns out that the coarse cohomological dimension $X\text{-dim}_G$ does not always agree with the asymptotic dimension even for proper metric spaces of bounded geometry and when the latter is finite (§5). Thus, an asymptotic analog of Alexandroff Theorem does not hold for this dimension. We recall that the most elegant argument for the Alexandroff theorem uses the representation theorem for the Čech cohomology. In view of our result the macro-micro analogy suggests that Roe's coarse cohomology theory lacks Brown's representability.

Moreover, it turns out that the asymptotic Morita theorem does not hold true for general proper metric spaces of bounded geometry. We construct a counterexample in §5 based on a new cohomological approach to asymptotic dimension and an idea from elementary number theory. In §6 we show how this example can be turned into a large scale fractal-like space. We present there a general construction of large simplicial complexes of prescribed shapes on all scales.

A new cohomological approach to the asymptotic dimension is based on cohomology groups defined by bounded cochains. Using bounded cohomology we define an asymptotic cohomological dimension $\text{asdim}_{\mathbf{Z}}$ with integral coefficients in §2 and we show that $\text{asdim}_{\mathbf{Z}} X = \text{asdim } X$ for proper metric spaces of bounded geometry with $\text{asdim } X < \infty$

in §3. So this approach has an asymptotic analog of the Alexandroff Theorem and it also gives a hope for developing an analog of the Bockstein theory.

For finite coefficients F the bounded cohomologies coincide with the standard cohomologies. We use this to show (§4) that $\text{asdim}_F Y = X - \dim_F Y$ in this case. In view of this equality, our approach to the asymptotic dimension still gives a hope for the formula $\text{asdim } \Gamma = cd(\Gamma)$ for torsion-free finitely presented groups with finite asymptotic dimension. Now in order to derive this formula it suffices to show that in the case of a finitely presented group Γ , $\text{asdim}_{\mathbf{Z}} \Gamma = \text{asdim}_{\mathbf{Z}_p} \Gamma$ for some p . We recall that in the world of compacta this holds true for all locally nice spaces [Dr3]. It is still unclear whether finitely presented groups are sufficiently nice among all discrete proper metric spaces.

§1 PRELIMINARIES

Asymptotic dimension. Gromov defined the asymptotic dimension of a metric space X as follows [Gr1].

DEFINITION. *A metric space X has asymptotic dimension $\leq n$ if, for every $d > 0$, there is an R and $n + 1$ d -disjoint, R -bounded families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X such that $\cup_{i=0}^n \mathcal{U}_i$ is a cover of X .*

We say that a family \mathcal{U} of subsets of X is R -bounded if $\sup\{\text{diam } U \mid U \in \mathcal{U}\} \leq R$. Also, \mathcal{U} is said to be d -disjoint if $d(x, y) > d$ whenever $x \in U, y \in V, U \in \mathcal{U}, V \in \mathcal{U}$, and $U \neq V$.

It is known that $\text{asdim } X \leq n$ if and only if X admits a uniformly bounded open cover \mathcal{U} of multiplicity $\leq n + 1$ with an arbitrary large Lebesgue number $L(\mathcal{U}) = \inf_{x \in X} \sup_{U \in \mathcal{U}} d(x, X \setminus U)$.

The notion of asymptotic dimension is a coarse invariant (see [Ro1] [Ro2]). Therefore the invariant $\text{asdim } \Gamma$ is well-defined for every finitely generated group (in fact for every countable group Γ [DS]).

A metric space X is called *proper* if every closed ball $B_r(x) \subset X$ is compact.

We recall that a cover \mathcal{V} of a space X is called a *refinement* of a cover \mathcal{U} , (the notation for this relation is $\mathcal{V} \prec \mathcal{U}$), if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$. Note that if the mesh of \mathcal{V} is less than the Lebesgue number $L(\mathcal{U})$ of \mathcal{U} , then $\mathcal{V} \prec \mathcal{U}$. A refinement defines a simplicial map of the nerves $q : N(\mathcal{V}) \rightarrow N(\mathcal{U})$ which is called a *refinement map*.

Let \mathcal{C} be a collection of sets in X and let $A \subset X$. The star of A with respect to \mathcal{C} is the set

$$St(A, \mathcal{C}) = \bigcup_{C \in \mathcal{C}, C \cap A \neq \emptyset} C.$$

If τ is the collection of simplices of a simplicial complex K , then for every vertex $v \in K$, $St(v, \tau) = St(v, K)$ is the combinatorial star of the vertex. The open star of a vertex $Ost(v, K) = Int(St(v, K))$ can be also defined as the complement to the link $St(v, K) \setminus Lk(v, K)$.

A cover \mathcal{V} of a space X is called a *star refinement* of a cover \mathcal{U} , $\mathcal{V} \prec \mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $St(V, \mathcal{V}) \subset U$. The corresponding simplicial map $q : N(\mathcal{V}) \rightarrow N(\mathcal{U})$ is called a *star refinement map*.

Anti-Čech approximation. A sequence of uniformly bounded locally finite open coverings $\{\mathcal{U}_i\}$ of a metric space X is called an *anti-Čech approximation* [Ro1] for X if $\mathcal{U}_i \prec \mathcal{U}_{i+1}$ for all i and the Lebesgue number $L(\mathcal{U}_i)$ tends to infinity. Let $N_i = N(\mathcal{U}_i)$ denote the nerve of \mathcal{U}_i . Then every anti-Čech approximation defines a direct system of locally finite simplicial complexes with refinement maps as the bonding maps:

$$N_1 \xrightarrow{q_2^1} N_2 \xrightarrow{q_3^2} \dots \rightarrow N_k \xrightarrow{q_{k+1}^k} N_{k+1} \rightarrow \dots .$$

For a locally finite open cover \mathcal{U} of a metric space X let $p : X \rightarrow N(\mathcal{U})$ denote a projection to the nerve defined by the partition of unity $\{\phi_U\}$ with $\phi_U^{-1}(0) = X \setminus U$. We call such projection *canonical* if it is defined by the following partition of unity:

$$\phi_U(x) = \frac{d(x, X \setminus U)}{\sum_V d(x, X \setminus V)}.$$

We recall that a cover \mathcal{U} is called *irreducible* if for every $U \in \mathcal{U}$ the family $\mathcal{U} \setminus \{U\}$ is not a cover. If the cover \mathcal{U} is irreducible then all vertices of $N(\mathcal{U})$ are in the image of p . For a map p with such property we say that it is *essentially surjective*. We always will assume that the covers \mathcal{U}_i in the definition of an anti-Čech approximation are irreducible.

A *uniform* metric on a simplicial complex K is the metric restricted from the Hilbert space $\ell_2(K^{(0)})$ under the natural realization of K . The geodesic metric on K induced from the uniform metric is called *uniform geodesic*. Usually we normalize this metric by $1/\sqrt{2}$ to have the length of every edge in all simplices equal one.

A map $f : X \rightarrow K$ of a metric space to a simplicial complex is called *uniformly cobounded* if there is $D > 0$ such that $\text{diam}(f^{-1}(\Delta)) \leq D$ for every simplex Δ .

Let $f : X \rightarrow Y$ be a map between metric spaces. We denote the number (if it exists)

$$Lip(f) = \sup\left\{\frac{d_Y(f(x), f(y))}{d_X(x, y)} \mid x, y \in X, x \neq y\right\}$$

and call it *the Lipschitz constant* of f . Every number $\lambda \geq Lip(f)$ is called *a Lipschitz constant* for f .

We note that for every irreducible uniformly bounded locally finite open cover \mathcal{U} of a metric space X with the Lebesgue number L the canonical projection $p : X \rightarrow N(\mathcal{U})$ to the nerve is c_n/L -Lipschitz and uniformly cobounded where n is the multiplicity of \mathcal{U} [BD1]. Moreover, it is a quasi-isometry provided X is geodesic and the nerve is taken with geodesic metric:

We recall that $p : X \rightarrow Y$ is a (l, D) -quasi-isometric embedding if there are constants $\lambda \geq 1$ and D such that

$$\frac{1}{l}d_X(x, y) - D \leq d_Y(p(x), p(y)) \leq ld_X(x, y) + D.$$

An (l, D) -quasi-isometric embedding $p : X \rightarrow Y$ is called a *quasi-isometry* if there is a constant r such that the image $p(X)$ is r -dense in Y , i.e. $N_r(p(X)) = Y$. We use the notations $N_r(A) = \{x \in X \mid d(x, A) \leq r\}$ and $ON_r(A) = \{x \in Y \mid d(x, A) < r\}$, $r > 0$ for closed and open r -neighborhoods of the set A in a metric space Y . Also we consider "negative" neighborhoods $N_{-r}(A) = Y \setminus N_r(X \setminus A)$, $r > 0$.

The following lemma can be extracted from [BD1] and Dr4].

Lemma 1.1. *Given n there is a constant C_n such that for every uniformly bounded cover \mathcal{U} of a proper geodesic metric space X with multiplicity of $\mathcal{U} \leq n + 1$ the canonical projection $p : X \rightarrow N(\mathcal{U})_{UG}$ to the nerve is a ϵ -Lipschitz $(l, 3)$ -quasi-isometry with $r = 1$ where $\epsilon = C_n/L(\mathcal{U})$ and $l = C_n \text{mesh}(\mathcal{U})$.*

Note that if the multiplicity of coverings \mathcal{U}_i in an anti-Čech approximation of X is bounded from above then the canonical projections $p_i : X \rightarrow N(\mathcal{U}_i)$ are ϵ_i -Lipschitz with $\lim \epsilon_i = 0$. Thus, a metric space X has $\text{asdim } X \leq n$ if and only if it admits an anti-Čech approximation $\mathcal{U}_1 \prec \mathcal{U}_2 \prec \mathcal{U}_3 \prec \dots$ with the multiplicity of \mathcal{U}_i bounded from above by $n + 1$. In terms of projections to the nerves it equals to the property that X admits a sequence $p_i : X \rightarrow N_i$ of uniformly cobounded ϵ_i -Lipschitz maps to uniform n -dimensional complexes with $\lim \epsilon_i = 0$.

We recall that a metric space X is said to be *of bounded geometry* (on large scale) if for every R the 1-capacity of R -balls in X is uniformly bounded from above. It was shown in [Ro1] that given $\lambda > 0$, every metric space of bounded geometry X admits an open cover of finite multiplicity with the Lebesgue number greater than λ . Thus every metric space of bounded geometry admits an anti-Čech approximation.

Let $p : K \rightarrow N$ be map between simplicial complexes. A simplicial map $q : K \rightarrow N$ is called a *simplicial approximation* of p if $p^{-1}(\Delta) \subset q^{-1}(\Delta)$ for every simplex $\Delta \subset N$. This is equivalent to the condition $p^{-1}(L) \subset q^{-1}(L)$ for every subcomplex $L \subset N$.

Proposition 1.2. *Every proper geodesic metric space X with $\text{asdim } X \leq n$ admits an anti-Čech approximation $\{\mathcal{U}_i, q_{i+1}^i\}$ with n -dimensional locally finite nerves N_i and essentially surjective projections $p_i : X \rightarrow N_i$ such that*

- (1) *there are bonding maps $p_{i+1}^i : N_i \rightarrow N_{i+1}$ with $p_{i+1} = p_{i+1}^i \circ p_i$ for all i ,*
- (2) *$(p_{i+1}^i)^{-1}(K)$ is a subcomplex for every subcomplex $K \subset N_{i+1}$,*
- (3) *simplicial maps $q_{i+1}^i : N_i \rightarrow N_{i+1}$ are simplicial approximations of p_{i+1}^i ,*
- (4) *$\text{Lip}(p_{i+1}^i) < 1/2$, and*
- (5) *$\mathcal{U}_i \prec \mathcal{U}_{i+1}$ for all i and q_{i+1}^i is a star refinement map.*

Proof. We construct these coverings and maps by induction. Assume that a sequence of covers $\mathcal{U}_1 \prec \mathcal{U}_2 \prec \dots \mathcal{U}_k$ together with the maps p_i , p_{i+1}^i , and q_{i+1}^i satisfying conditions (1)-(5) is constructed. We assume that p_1 and all p_{i+1}^i are canonical projections to the nerves. By Proposition 1.1 N_k is quasi-isometric to X and hence $\text{asdim } N_k \leq n$. We consider a uniformly bounded cover \mathcal{V}' of N_k of multiplicity $n+1$ with the Lebesgue number $\geq 2C_n + 2 \geq 5$. Then $N_{-1}(\mathcal{V}') = \{N_{-1}(V') \mid V' \in \mathcal{V}'\}$ has the Lebesgue number $\geq 2C_n + 3$. We define $\mathcal{V} = \{Int(St(V, N_k)) \mid V \in N_{-1}(\mathcal{V}')\}$. Then $\mathcal{V} \prec \mathcal{V}'$ and hence, the nerve of \mathcal{V} is n -dimensional and $\{Ost(v, N_k) \mid v \in N_k^{(0)}\} \prec \mathcal{V}$. Define $\mathcal{U}_{k+1} = p_k^{-1}\mathcal{V}$. Then

$$\mathcal{U}_k = p_k^{-1}\{Ost(v, N_k) \mid v \in N_k^{(0)}\} \prec p_k^{-1}\mathcal{V} = \mathcal{U}_{k+1}.$$

Let $N_{k+1} = N(\mathcal{V}) = N(\mathcal{U}_{k+1})$, let $p_{k+1}^k : N_k \rightarrow N_{k+1}$ be the canonical projection to the nerve, and let $p_{k+1} = p_{k+1}^k p_k$. We define $q_{k+1}^k(v)$ to be an element $Int(St(V, N_k))$ of \mathcal{V} , $V \in \mathcal{V}'$ such that $St(v) \subset V$. By Proposition 1.2 p_{k+1}^k is $C_n/L(\mathcal{V})$ -Lipschitz. Thus, $\text{Lip}(p_{k+1}^k) \leq 1/2$ by the choice of \mathcal{V} . Thus the conditions (1), (4), and (5) are satisfied.

To verify (2) it suffices to show that the preimage $(p_{k+1}^k)^{-1}(\Delta)$ is a subcomplex in N_k for every simplex Δ in $N(\mathcal{V})$. Note that

$$(p_{k+1}^k)^{-1}(\Delta) = N_k \setminus \bigcup_{W \notin \Delta^{(0)}} W = N_k \setminus \bigcup_W Int(K_W) = \bigcap_W (N_k \setminus Int K_W)$$

where $K_W = St(V_W, N_k)$ is a subcomplex of N_k . Hence $(p_{k+1}^k)^{-1}(\Delta)$ is a subcomplex of N_k as the intersection of subcomplexes.

Finally we check (3). Let $p_{k+1}^k(z) \in \Delta = [V_1, \dots, V_j]$. Then if $z \in V \in \mathcal{V}$, it follows that V equals one of V_i , $i \leq j$. Let $z \in [v_1, \dots, v_s] \subset N_k$. By the definition $z \in St(v_i) \subset q_{k+1}^k(v_i)$. Therefore, $q_{k+1}^k(v_i) \in \Delta^{(0)}$. Hence $q_{k+1}^k(z) \in \Delta$. \square

We will refer to such anti-Čech approximation as to *regular* and will denote it as

$$\{p_i : X \rightarrow N_i, q_{i+1}^i, p_{i+1}^i\}$$

where p_i are ϵ_i -Lipschitz with $\lim \epsilon_i = 0$. In some instances we will not assume the condition (5). Note that the coverings \mathcal{U}_i can be recovered from this data as $\mathcal{U}_i = p_i^{-1}(Ost(v, N_i))$ where $Ost(v, N)$ is the open star of a vertex v in a simplicial complex N .

§2 DEFINITION OF $\text{asdim}_{\mathbf{Z}}$

Bounded cohomology. Let K be a simplicial complex. An integral cochain $\phi : C_m(K) \rightarrow \mathbf{Z}$ is bounded if there is a constant $b < \infty$ such that $|\phi(\sigma)| < b$ for all m -simplices σ in K . Clearly, the coboundary $\delta\phi$ of a bounded cochain is a bounded cochain. Cohomology groups defined by means of bounded cochains C_b^* are called *bounded cohomologies* of K and denoted as $H_b^*(K)$. Clearly, the inclusion $C_b^* \rightarrow C^*$ is a chain map. Therefore there is a natural homomorphism $H_b^*(K) \rightarrow H^*(K)$. Every simplicial map $\phi : K \rightarrow L$ induces a homomorphism $H_b^*(L) \rightarrow H_b^*(K)$. This definition can be extended to any coefficient group with a norm such as \mathbf{Q} or \mathbf{R} and their subgroups. For a subcomplex $L \subset K$ one can define a relative bounded cohomology groups $H_b^*(K, L)$ by considering the relative cochains. We note that for bounded cohomology there are the exact sequence of pair

$$\dots \leftarrow H_b^i(L) \leftarrow H_b^i(K) \leftarrow H_b^i(K, L) \leftarrow H_b^{i-1}(L) \leftarrow \dots,$$

exact sequence of triple

$$\dots \leftarrow H_b^i(L, A) \leftarrow H_b^i(K, A) \leftarrow H_b^i(K, L) \leftarrow H_b^{i-1}(L, A) \leftarrow \dots$$

where $A \subset L$ is a subcomplex, and the excision isomorphism

$$H_b^i(K, B) \xrightarrow{\cong} H_b^i(A, A \cap B)$$

where $A, B \subset K$ are subcomplexes such that $K = A \cup B$.

Approximation by asymptotic polyhedra. Let X be a metric space. We consider locally finite covers \mathcal{U} of X by bounded open sets such that the Lebesgue number $L_x(\mathcal{U})$ tends to infinity as $x \rightarrow \infty$. We say that \mathcal{V} is a refinement of \mathcal{U} at infinity if there is $R > 0$ such that for every $V \in \mathcal{V}$, $d(V, x_0) > R$, there is $U \in \mathcal{U}$ with $V \subset U$. A refinement at infinity $\mathcal{V} \prec_{\infty} \mathcal{U}$ defines a simplicial map $\phi : K_{\mathcal{V}} \rightarrow K_{\mathcal{U}}$ between corresponding subcomplexes of the nerves. All such covers with the relation \prec_{∞} form a directed set $aCov(X)$.

The following proposition is proven in [Dr1].

Proposition 2.1. *If a metric space X has $\text{asdim } X \leq n$ then the family $a\text{Cov}(X)$ has a cofinal subfamily $a\text{Cov}_n(X)$ that consists of covers with n -dimensional nerves.*

A countable simplicial complex K with a metric d such that every simplex is isometric to an affine simplex in a Hilbert space is called an *asymptotic polyhedron* if $\lim_{i \rightarrow \infty} \text{Lip}(\phi_i) = 0$ where K is the union of simplices $K = \bigcup_i \sigma_i$ and $\phi_i : \sigma_i \rightarrow \Delta^{\dim \sigma_i}$ is the affine map to the standard simplex.

We note that if a cover \mathcal{U} has bounded multiplicity then its nerve N admits a metric of an asymptotic polyhedron such that the projection $p_{\mathcal{U}} : X \rightarrow N$ is 1-Lipschitz (see [Dr1] [DFW]).

Asymptotic bounded cohomology. Let K be a simplicial complex. Denote by $\mathcal{F}(K)$ the set of all finite subcomplexes of K . We define the *bounded cohomology at infinity* $AH_b^i(K)$ of a complex K in the dimension i as the direct limit

$$AH_b^i(K) = \varinjlim \{H_b^i(K \setminus F) \mid F \subset K, F \in \mathcal{F}(K)\}.$$

Let X be a metric space and let $\mathcal{V} \prec_{\infty} \mathcal{U}$ be covers as above. Then there is a well-defined homomorphism $AH_b^*(N(\mathcal{U})) \rightarrow AH_b^*(N(\mathcal{V}))$ between the bounded cohomology at infinity of nerves.

For a metric space X we define its *asymptotic bounded cohomology* as the direct limit

$$AH^i(X) = \varinjlim \{AH_b^i(N(\mathcal{U})) \mid \mathcal{U} \in a\text{Cov}(X)\}.$$

Thus, it can be defined as

$$AH^i(X) = \varinjlim \{H_b^i(N(\mathcal{U}) \setminus F) \mid \mathcal{U} \in a\text{Cov}(X), F \in \mathcal{F}(N(\mathcal{U}))\}.$$

This definition can be extended to any coefficient group with the norm.

Let L be a subcomplex of a simplicial complex K . We define the bounded cohomology at infinity $AH_b^*(K, L)$ of the pair (K, L) in the dimension i as the direct limit

$$AH_b^i(K, L) = \varinjlim \{H_b^i(K \setminus F, L \setminus F) \mid F \subset K, F \in \mathcal{F}\}.$$

Let X be a metric space and let $Y \subset X$ be a subset. For a cover \mathcal{U} of X we denote by $N(\mathcal{U}|_Y)$ the nerve of the cover \mathcal{U} restricted to Y , $\mathcal{U}|_Y = \{U \cap Y \mid U \in \mathcal{U}\}$. Clearly, $N(\mathcal{U}|_Y) \subset N(\mathcal{U})$ and a refinement $\mathcal{V} \prec \mathcal{U}$ defines a simplicial map of pairs $(N(\mathcal{V}), N(\mathcal{V}|_Y)) \rightarrow (N(\mathcal{U}), N(\mathcal{U}|_Y))$. Then we define a relative asymptotic cohomology as the direct limit

$$AH^i(X, Y) = \varinjlim \{AH_b^i(N(\mathcal{U}), N(\mathcal{U}|_Y)) \mid \mathcal{U} \in a\text{Cov}(X)\}.$$

Let $Y \subset X$ be a subset of a metric space. For a cover \mathcal{U} of X we denote by $\mathcal{U}_Y = \{U \in \mathcal{U} \mid U \cap Y \neq \emptyset\}$. By $aCov_X(Y)$ we denote the set of locally finite covers \mathcal{V} of Y by bounded open sets from X with $\lim_{x \rightarrow \infty, x \in Y} L_x(\mathcal{V}) = \infty$. Thus, for every $\mathcal{U} \in aCov(X)$ we have $\mathcal{U}_Y \in aCov_X(Y)$. Note that every cover $\mathcal{V} \in aCov_X(Y)$ can be enlarged to a cover $\mathcal{U} \in aCov(X)$, $\mathcal{V} \subset \mathcal{U}$, in such a way that $\mathcal{U}|_Y = \mathcal{V}|_Y$.

Proposition 2.2. *The family $\{\mathcal{U}|_Y \mid \mathcal{U} \in aCov(X)\} = \{\mathcal{U}|_Y \mid \mathcal{U} \in aCov_X(Y)\}$ is cofinal in $aCov(Y)$ where $Y \subset X$ is taken with the restriction metric.*

Proof. Let $\mathcal{W} \in aCov(Y)$. For every $W \in \mathcal{W}$ we define $\tilde{W} = \cup_{x \in W} B_{r_x/3}(x)$ where r_x is the supremum of r such that $B_r(x) \cap Y \subset W$ and $B_r(x)$ is the r -ball in X . Clearly, $\tilde{W} \cap Y = W$. Let $\tilde{\mathcal{W}} = \{\tilde{W} \mid W \in \mathcal{W}\}$. We note the Lebesgue number of $\tilde{\mathcal{W}}$ restricted to Y tends to infinity. Thus, $\tilde{\mathcal{W}} \in aCov_X(Y)$ \square

REMARK. Note that $N(\tilde{\mathcal{W}}) = N(\mathcal{W})$. Since $\{\tilde{\mathcal{W}} \mid \mathcal{W} \in aCov(Y)\}$ is cofinal in $aCov_X(Y)$, we obtain

$$AH^*(Y) = \lim_{\rightarrow} \{AH_b^*(N(\mathcal{U}|_Y)); \mathcal{U} \in aCov(X)\} = \lim_{\rightarrow} \{AH_b^*(N(\mathcal{U}_Y)); \mathcal{U} \in aCov(Y)\}.$$

Thus, the asymptotic cohomology of a pair can be defined as

$$AH^i(X, Y) = \lim_{\rightarrow} \{AH_b^i(N(\mathcal{U}), N(\mathcal{U}_Y)) \mid \mathcal{U} \in aCov(X)\}.$$

Since bounded cohomologies at infinity posses the exactness, the exactness is preserved by direct limits, and in view of Proposition 2.2, there are exact sequences for asymptotic cohomology of pair (and triple).

A triad $X, A, B, A \cup B = X$, is called *excisable* if the family $\{\mathcal{U}_A \cap \mathcal{U}_B \mid \mathcal{U} \in aCov(X)\}$ is cofinal in $aCov_X(A \cap B)$.

Proposition 2.3. *For every excisable triad X, A, B there is the Mayer-Vietoris exact sequence*

$$\dots \rightarrow AH^n(A) \oplus AH^n(B) \rightarrow AH^n(X) \rightarrow AH^{n+1}(A \cap B) \rightarrow \dots$$

Proof. We note that the Mayer-Vietoris sequence holds for bounded cohomology and hence for bounded cohomology at infinity for simplicial complexes. We consider the direct limit of these Mayer-Vietoris sequences for the nerves $N(\mathcal{U})$, $\mathcal{U} \in aCov(X)$:

$$\rightarrow AH_b^n(N(\mathcal{U}_A)) \oplus AH_b^n(N(\mathcal{U}_B)) \rightarrow AH_b^n(N(\mathcal{U})) \rightarrow AH_b^{n+1}(N(\mathcal{U}_A) \cap N(\mathcal{U}_B)) \rightarrow \dots$$

Note that there are the inclusions $N(\mathcal{U}_A) \cap N(\mathcal{U}_B) = N(\mathcal{U}_A \cap \mathcal{U}_B)$. Since $\{\mathcal{U}_A \cap \mathcal{U}_B \mid \mathcal{U} \in aCov(X)\}$ is cofinal in $aCov_X(A \cap B)$, one can argue that

$$\lim_{\rightarrow} AH_b^*((N(\mathcal{U}|_A) \cap N(\mathcal{U}|_B)) = \lim_{\rightarrow} AH_b^*(N(\mathcal{U}|_{A \cap B})) = AH^*(A \cap B).$$

□

There is a version of the Mayer-Vietoris sequence for pairs.

Proposition 2.4. *For every excisable triad $A \cup B, A, B$ of subsets in a metric space X there is the Mayer-Vietoris exact sequence*

$$\dots \rightarrow AH^n(X, A) \oplus AH^n(X, B) \rightarrow AH^n(X, A \cap B) \rightarrow AH^{n+1}(X, A \cup B) \rightarrow \dots .$$

A metric space X is called *uniformly path connected* if there is a monotone tending to infinity continuous function $S : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, $S(0) = 0$, such that every two points $x, x' \in X$ can be joined by a path J with $diam(J) \leq S(d(x, x'))$. We note that every geodesic metric space is uniformly path connected.

Proposition 2.5. *Let $W \subset X$ be an open subset of a uniformly path connected metric space X . Then the triple $X, \bar{W}, X \setminus W$ is excisable.*

Proof. Let $\mathcal{V} \in aCov_X \partial W$. Let $\mathcal{U}' \in aCov(X)$ be an enlargement of \mathcal{V} , $\mathcal{V} \subset \mathcal{U}'$, and $\mathcal{U}'|_Y = \mathcal{V}|_Y$. Since X is uniformly path connected, the cover \mathcal{U} that consists of components of sets from \mathcal{U}' belongs to $aCov(X)$. Then for every $U \in \mathcal{U}$ with $U \cap \bar{W} \neq \emptyset$ and $U \cap (X \setminus W) \neq \emptyset$, we obtain $U \cap \partial W \neq \emptyset$. Thus, U is a component of \mathcal{U}' with $U' \in \mathcal{V}$. We have checked that $\mathcal{U}|_{\bar{W}} \cap \mathcal{U}|_{X \setminus W} \prec \mathcal{V}$. □

Proposition 2.6. *Let $W \subset X$ be an open subset of a uniformly path connected metric space X . Then there is the excision isomorphism*

$$AH^n(X, X \setminus W) = AH^n(\bar{W}, \partial W).$$

Proof. Let $\mathcal{U} \in aCov(X)$. Note that there is an inclusion $\partial N(\mathcal{U}_{\bar{W}}) \subset N(\mathcal{U}_{\bar{W}} \cap \mathcal{U}_{X \setminus W})$ which is essentially onto. Moreover, if $\mathcal{V} \prec \mathcal{U}$, then there is a map $\xi : N(\mathcal{V}_{\bar{W}} \cap \mathcal{V}_{X \setminus W}) \rightarrow \partial N(\mathcal{U}_{\bar{W}})$ that make the diagram generated by ξ and the refinement map commutative. This implies that

$$\lim_{\rightarrow} AH_b^*(\partial N(\mathcal{U}_{\bar{W}})) \rightarrow \lim_{\rightarrow} AH_b^*(N(\mathcal{U}_{\bar{W}} \cap \mathcal{U}_{X \setminus W}))$$

is an isomorphism. In view of Proposition 2.5 the later limit is equal to $\lim_{\rightarrow} AH_b^*(N(\mathcal{U}_{\partial W}))$. Therefore,

$$\lim_{\rightarrow} AH_b^*(N(\mathcal{U}_{\bar{W}}), \partial N(\mathcal{U}_{\bar{W}})) = \lim_{\rightarrow} AH_b^*(N(\mathcal{U}_{\bar{W}}), N(\mathcal{U}_{\partial W})) = AH^*(\bar{W}, \partial W).$$

By the excision for bounded cohomology at infinity of simplicial complexes we obtain

$$AH^*(X, X \setminus W) = \lim_{\rightarrow} AH_b^*(N(\mathcal{U}), N(\mathcal{U}_{X \setminus W})) = \lim_{\rightarrow} AH_b^*(N(\mathcal{U}_{\bar{W}}), \partial N(\mathcal{U}_{\bar{W}})).$$

□

Asymptotic cohomological dimension. Let X be a metric space we define its asymptotic integral cohomological dimension as follows:

$$\text{asdim}_{\mathbf{Z}} X = \max\{n \mid AH^n(A, B) \neq 0 \mid B \subset A \subset X\}.$$

This definition can be extended to any coefficient group G with a semi-norm. The notation is $\text{asdim}_G X$ when the semi-norm on G is specified.

REMARK. One can show that like in the case of cohomological dimension in topology it suffices to consider $A = X$ in the above definition. Indeed, from exact sequence of triple $B \subset A \subset X$

$$AH^{n+1}(X, A) \leftarrow AH^n(A, B) \leftarrow AH^n(X, B) \leftarrow AH^n(X, A) \leftarrow$$

it follows that if $AH^n(A, B) \neq 0$ for $n = \text{asdim}_{\mathbf{Z}} X$, then $AH^{n+1}(X, A) = 0$ and hence $AH^n(X, B) \neq 0$.

§3 CONNECTION BETWEEN $\text{asdim}_{\mathbf{Z}}$ AND asdim

In this section we show that $\text{asdim}_{\mathbf{Z}}$ agrees with asdim for proper geodesic metric spaces provided the later is finite.

Proposition 3.1. *Let $K \subset N$ be a subcomplex of a uniform geodesic complex of dimension n . Suppose that $f : K \rightarrow L$ is the map to an b -bounded metric space such that $\text{Lip}(f|_{\sigma}) \leq \lambda$, $\lambda \geq 1$, for all simplices $\sigma \subset K$. Then f is $c\lambda$ -Lipschitz where c depends on n and b only.*

Proof. Let K be realized in the Hilbert space $\ell_2(K^{(0)})$ spanned by the vertices of K . Then the identity map $1_K : (K, d_N) \rightarrow (K, d_{\ell_2})$ is 1-Lipschitz. It suffices to show that f

is $c\lambda$ -Lipschitz with respect to the Hilbert space metric. Note that the distance between two disjoint simplices in $\Delta \subset \ell_2$ of dimension $\leq n$ is at least $\sqrt{\frac{2}{n+1}}$. For every chain $0 \leq l \leq k \leq k'$ we denote by $D_{l,k,k'}$ the union of standard simplices $\Delta^k \cup \Delta^{k'}$ in ℓ_2 with the intersection $\Delta^k \cap \Delta^{k'} = \Delta^l$. Let $c_{l,k,k'}$ be the Lipschitz constant of the identity map $id : (D_{l,k,k'}, d_{\ell_2}) \rightarrow (D_{l,k,k'}, d_{geod})$ where d_{geod} is the intrinsic metric on $D_{l,k,k'}$ induced by the Euclidean metric. Let $\bar{c} = \max\{c_{l,k,k'} \mid k' \leq 2n+1\}$. We take $c = b\sqrt{n+1}\bar{c}$. Then for every couple of points $x, x' \in K$ with $\|x - x'\| \geq \sqrt{\frac{2}{n+1}}$ we obtain $d_L(f(x), f(x')) \leq b \leq c\|x - x'\| \leq c\lambda\|x - x'\|$. Assume that $x \in \Delta^k \subset K$, $x' \in \Delta^{k'} \subset K$, and $\|x - x'\| < \sqrt{\frac{2}{n+1}}$ where $\Delta^k, \Delta^{k'}$ are simplices. Then $\Delta^k \cap \Delta^{k'} \neq \emptyset$. Hence there is a geodesic segment J in $\Delta^k \cup \Delta^{k'}$ joining x with x' of length $|J|$. We may assume that it is piece-wise linear (actually it consists of two straight intervals). Since f is λ -Lipschitz on each of the segments of J , we obtain $d_L(f(x), f(x')) \leq \lambda|J| \leq \lambda c\|x - x'\|$. \square

Let $f : X \rightarrow \Delta$ be a map of a metric space to the standard simplex. Denote by

$$\delta(f) = \inf\{Lip(\psi) \mid \psi : X \rightarrow \partial\Delta, \psi|_{f^{-1}(\partial\Delta)} = f|_{f^{-1}(\partial\Delta)}\}.$$

Let X be a proper geodesic metric space with an anti-Čech approximation $\{p_i : X \rightarrow N_i\}$ where $\dim N_i = n$ for all i . For every i we define

$$\delta^i = \overline{\lim_{\Delta \rightarrow \infty}} \delta(p_i|_{p_i^{-1}(\Delta)})$$

where Δ runs over all n -simplices from N_i .

Proposition 3.2. *Suppose that $\text{asdim } X = n$ for a proper geodesic metric space X and let $\{p_i : X \rightarrow N_i\}$ to be an anti-Čech approximation of X by n -dimensional polyhedra. Then there is $c > 0$ such that $\delta^i > c$ for all i .*

Proof. Assume the contrary. By passing to a subsequence we may assume that $\delta^i \rightarrow 0$. Let $C_i \subset N_i$ be a finite subcomplex such that $\delta(p_i|_{p_i^{-1}(\Delta)}) < 2\delta^i$ for all n -simplices $\Delta \subset N_i \setminus C_i$. Then by sweeping one can define a map $\xi_i : X \rightarrow N_i^{(n-1)} \cup C_i$ which is $2\delta^i$ -Lipschitz on every set of the from $p_i^{-1}(\Delta)$. Since X is geodesic, the map ξ_i is $2\delta^i$ -Lipschitz where $N_i^{(n-1)} \cup C_i$ is taken with restricted metric from N_i and N_i is supplied with the uniform geodesic metric. Consider the quotient map $q_i : N_i^{(n-1)} \cup C_i \rightarrow N_i^{(n-1)} \cup C_i / C_i = K_i$ and take a uniform bounded metric on $K_i \subset \ell_2(K^{(0)})$. By Proposition 3.1 q_i is c -Lipschitz where c depends on n only. Thus, $\psi_i = q_i \circ \xi_i : X \rightarrow K_i$ is $2c\delta^i$ -Lipschitz.

Therefore, $\{\psi_i : X \rightarrow K_i\}$ is an anti-Čech approximation $\{\psi_i : X \rightarrow K_i\}$ of X by $n-1$ -dimensional polyhedra. Then by the definition $\text{asdim } X \leq n-1$ (see the discussion after Lemma 2.1), which contradicts to the assumption of the Proposition. \square

The following lemma is taken from [Dr4], Lemma 2.2.

Lemma 3.3. *Suppose that X and Y are finite uniform simplicial complexes. Then for every λ there exists $\mu = \mu(\lambda)$ such that every null homotopic λ -Lipschitz map $f : X \rightarrow Y$ admits a μ -Lipschitz homotopy $H : X \times I \rightarrow Y$ to a constant map.*

Here we give a short review of the classical extension theory. Let (K, A) be a CW complex pair. Suppose that an extension problem

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \subset \downarrow & & \\ K & & \end{array}$$

for a simply connected space Y is resolved on the n -skeleton $K^{(n)}$ of K by a map $g : A \cup K^{(n)} \rightarrow Y$. The map g defines the obstruction cochain $c_g : C_{n+1}(K, A) \rightarrow \pi_n(Y)$ which is a relative cocycle. If a corresponding relative cohomology class is zero, i.e., if $c_g = \delta\gamma$ for some γ , then one can change the map g on the relative n -skeleton without changing it on the $n-1$ -dimensional skeleton such that a new map g' can be extended to a map $\bar{g} : A \cup K^{(n+1)} \rightarrow Y$ [Hu].

If the homotopy group $\pi_n(Y)$ is supplied with a norm and Y is given some metric one can bring a quantitative statement.

Proposition 3.4. *Suppose that K is a simplicial complex supplied with a uniform metric and let $Y = S^n$ be the unit n -sphere. Suppose that in the above extension problem g is λ -Lipschitz and $|\gamma| \leq b$. Then g' can be taken μ' -Lipschitz and \bar{g} with a μ -Lipschitz restriction to every $n+1$ -simplex where μ' , $\bar{\mu}$ depend on n , λ and b only.*

Proof. We fix a triangulation on S^n , say, by identifying S^n with the boundary of the standard $n+1$ -simplex. Let $\sigma \subset S^n$ be an n -face. Using a simplicial approximation we may assume that g is simplicial with respect to some iterated barycentric subdivision $\beta^i(A \cup K^{(n)})$ where i depends on λ (and n). The Obstruction Theory [Hu] prescribes a construction of g' as a map having degree on each n -simplex σ' in $K \setminus A$ equal to $\deg(g|_{\sigma'}) - \gamma(\sigma')$ where the degree of $g|_{\sigma'}$ (as well as the degree of $g'|_{\sigma'}$) is computed for the map of pairs $g|_{\sigma'} : (\sigma', \partial\sigma') \rightarrow (S^n, S^n \setminus \text{Int}\sigma)$. Since $|\deg(g|_{\sigma'})|$ is bounded by the number of simplices in $\beta^i(\Delta^n)$. Thus, $|\deg(g|_{\sigma'}) - \beta(\sigma')|$ is uniformly bounded by

a number that depends only on n, λ and b . Since every σ' is isometric to the standard n -simplex, this degree can be realized by a μ' -Lipschitz map g' where μ' depends on n, λ and b only. According to Lemma 3.3 the extension \bar{g} of g' can be taken to be $\bar{\mu}$ -Lipschitz on every $n + 1$ -dimensional simplex of K . \square

Let $\sqcup \Delta_i^n$ be a disjoint union of (oriented) n -simplices. Then the n -cochain μ that take each simplex to 1 defines a nonzero element $\mu \in AH_b^n(\sqcup \Delta_i^n, \sqcup \partial \Delta_i^n)$ which we call *the fundamental class*.

The family $\{A_i\}$ of bounded subsets in a metric space X is called *dispersed* if there is a function $s : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ tending to infinity such that the family $\{A_i \setminus B_R(x_0)\}$ is $s(R)$ -disjoint.

Lemma 3.5. *Suppose that $\text{asdim } X = n$ for a proper geodesic metric space X and let $\{p_i : X \rightarrow N_i; p_{i+1}^i, q_{i+1}^i\}$ be a regular anti-Čech approximation of X by n -dimensional simplicial complexes. Then there is a sequence of n -simplices $\Delta_i \subset N_i$ such that the cohomology homomorphism induced by $\phi = \sqcup p_i|_{p_i^{-1}(\Delta_i)}$,*

$$\phi^* : AH_b^n(\sqcup \Delta_i, \sqcup \partial \Delta_i) \rightarrow AH^n(\sqcup p_i^{-1}(\Delta_i), \sqcup p_i^{-1}(\partial \Delta_i)),$$

takes the fundamental class μ to a nonzero element.

Proof. In view of Proposition 3.2 we can take $\Delta_i \subset N_i$ such that $\delta(p_i|_{p_i^{-1}(\Delta_i)}) > c/2$ and $\{p_i^{-1}(\Delta_i)\}$ is dispersed.

Denote by $M_i^k = (p_i^k)^{-1}(\Delta_i)$, and $K_i^k = (p_i^k)^{-1}(\partial \Delta_i)$, $k < i$. According to the condition (2) of a regular anti-Čech approximation M_i^k is a subcomplex of N_k . For every function $\kappa : \mathbf{N} \rightarrow \mathbf{N}$ with $\kappa = \kappa(i) < i$ and $\kappa(i) \rightarrow \infty$ we define an open cover \mathcal{U}_κ of $\sqcup p_i^{-1}(\Delta_i)$ as follows

$$\mathcal{U}_\kappa = \{p_{\kappa(i)}^{-1}(Ost(v, M_i^{\kappa(i)})) \mid v \in (M_i^{\kappa(i)})^{(0)}, i \in \mathbf{N}\}$$

where $Ost(v, M)$ denotes the open star of a vertex v in a complex M . Let $\{\mathcal{U}_i\}$ be the family of open covers that forms the above anti-Čech approximation. It is easy to check that for any j and for every subcomplex $L \subset N_j$, $p_j^{-1}(Ost(u, L)) = U \cap p_j^{-1}(L)$ where $U \in \mathcal{U}_j$ and u a vertex in the nerve N_j that corresponds to U . Then $\mathcal{U}_\kappa = \cup_i \mathcal{U}_{\kappa(i)}|_{p_i^{-1}(\Delta_i)}$. Then the nerve of \mathcal{U}_κ coincides with $\sqcup_i M_i^{\kappa(i)}$. Being the composition of simplicial approximations the map q_i^k is a simplicial approximation of p_i^k . Thus, we obtain $q_i^k(M_i^k) = \Delta_i$ for all $k < i$.

Since $\{p_i^{-1}(\Delta_i)\}$ is dispersed, for every $\mathcal{U} \in aCov(\sqcup p_i^{-1}(\Delta_i))$ we obtain that the Lebesgue number of the restrictions tends to infinity: $L(\mathcal{U}|_{p_i^{-1}(\Delta_i)}) \rightarrow \infty$. From here it is easy to verify that the family of covers

$$\{\mathcal{U}_\kappa \mid \kappa : \mathbf{N} \rightarrow \mathbf{N}, \kappa = \kappa(i) < i, \kappa(i) \rightarrow \infty\}$$

is cofinal in $aCov(\sqcup p_i^{-1}(\Delta_i))$. Note that $N(\mathcal{U}_{\kappa(i)}|_{p_i^{-1}(\partial\Delta_i)}) = K_i^{\kappa(i)}$.

Assume that $\phi^*(\mu) = 0$. Then there is $\kappa : \mathbf{N} \rightarrow \mathbf{N}$, $\kappa = \kappa(i) < i$ and $\kappa(i) \rightarrow \infty$ such that the homomorphism ψ^* induced by the simplicial map

$$\psi = \sqcup q_i^{\kappa(i)}|_{M_i^{\kappa(i)}} : \sqcup(M_i^{\kappa(i)}, K_i^{\kappa(i)}) \rightarrow \sqcup(\Delta_i, \partial\Delta_i)$$

takes μ to zero. Let $\psi_i = q_i^{\kappa(i)}|_{M_i^{\kappa(i)}}$. Denote by μ_i the image of the cocycle $1_i \in C^n(\Delta_i, \partial\Delta_i) \cong \mathbf{Z}$ that takes Δ_i to 1 under ψ_i^* . If $\phi_1^*(\mu) = 0$ then for all sufficiently large i there are a number $b > 0$ and b -bounded $n-1$ -dimensional cochains $\gamma_i \in C^{n-1}(M_i^{\kappa(i)})$ such that $\delta\gamma_i = \mu_i$. Since $q_i^{\kappa(i)}$ is a simplicial approximation of $p_i^{\kappa(i)}$, we obtain that $\psi_i(K_i^{\kappa(i)}) \subset \partial\Delta_i$. We consider the extension problem:

$$\begin{array}{ccc} K_i^{\kappa(i)} & \xrightarrow{\psi_i|_{\dots}} & \partial\Delta \\ j \downarrow & & \\ M_i^{\kappa(i)} & & \end{array}$$

where $j : K_i^{\kappa(i)} \rightarrow M_i^{\kappa(i)}$ is the inclusion. The map ψ_i solves this problem on the $n-1$ -skeleton and the relative cocycle μ_i is the obstruction cocycle to the solution of the problem on $M_i^{\kappa(i)}$. Since the obstruction cocycle is a coboundary, by Proposition 3.4 one can change the map ψ_i on the $(n-1)$ -skeleton of $M_i^{\kappa(i)} \setminus K_i^{\kappa(i)}$ without changing it on the $(n-2)$ -skeleton such that new map has an extension to $M_i^{\kappa(i)}$. Moreover, we may assume that there is $\bar{\lambda}$ such that for every i there is an extension $\bar{\psi}_i : M_i^{\kappa(i)} \rightarrow \partial\Delta_i$ which is $\bar{\lambda}$ -Lipschitz on every n -simplex of $M_i^{\kappa(i)}$. By Proposition 3.1 we can conclude that $\bar{\psi}_i$ is $c'\bar{\lambda}$ -Lipschitz on $M_i^{\kappa(i)}$ for all i . Note that

$$c/2 < \delta(\partial_i|_{p_i^{-1}(\Delta_i)}) \leq \text{Lip}(\bar{\psi}_i \circ p_{\kappa(i)}) < c'\bar{\lambda}\epsilon_{\kappa(i)}.$$

This contradict to the fact that $\epsilon_{\kappa(i)} \rightarrow 0$. \square

Asymptotic Alexandroff Theorem. The classical Alexandroff Theorem states that $\dim X = \dim_{\mathbf{Z}} X$ for compact metric spaces provided $\dim X < \infty$. Here we prove the following.

Theorem 3.6. *Suppose $\text{asdim } X < \infty$ for a proper geodesic metric space X . Then $\text{asdim}_{\mathbf{Z}} X = \text{asdim } X$.*

Proof. The inequality $\text{asdim}_{\mathbf{Z}} X \leq \text{asdim } X$ follows from Proposition 2.1 and the definition of $\text{asdim}_{\mathbf{Z}} X$.

Assume that $\text{asdim } X = n$. Then by Lemma 3.5 $AH^n(Z, Y) \neq 0$ for some $Y \subset Z \subset X$. Therefore, $\text{asdim}_{\mathbf{Z}} X \geq n$. \square

Lemma 3.7. *Suppose $\text{asdim } X < \infty$ for a uniformly path connected metric space X and let $\text{asdim}_G X = n$ for some abelian group G . Then there exists a dispersed family $\{U_i\}$ of bounded open sets in X such that $AH^n(\sqcup \bar{U}_i, \sqcup \partial U_i; G) \neq 0$.*

Proof. Let $\text{asdim } X = n$ and let $W \subset X$ be an open subset such that $AH^n(X, X \setminus W; G) \neq 0$. There are dispersed families of bounded open sets $\mathcal{V}^k = \{U_i^k\}$, $k = 0, \dots, n$, such that the union $\mathcal{V} = \cup_{k=0}^n \mathcal{V}^k$ is a cover $\mathcal{V} \in aCov(X)$ (see [DKU] for the construction). Denote by $\mathcal{U}^k = \mathcal{V}^k|_W$. Using induction and Propositions 2.4 and 2.5 we can derive that $AH^n(X, X \setminus \cup_{U \in \mathcal{U}^k} U; G) \neq 0$ for some k . We take \mathcal{U}^k as the desired family $\{U_i\}$. The excision (Proposition 2.6) implies that $AH^n(\sqcup \bar{U}_i, \sqcup \partial U_i; G) \neq 0$. \square

§4 RELATION TO THE COARSE COHOMOLOGICAL DIMENSION

In this section we consider only metric spaces Y that admit an anti-Čech approximation. Using an anti-Čech approximation $\{N_i, q_{i+1}^i\}$ of Y John Roe [Ro1] defined coarse cohomology group $HX^*(Y)$ of Y as the homology of the inverse limit of the cochain complexes $C_0^*(N_i)$ that consist of cochains with compact supports. Then the standard argument shows that coarse cohomology groups fit in the short exact sequence [Ro1]:

$$0 \rightarrow \lim^1 H_c^{k-1}(N_i) \rightarrow HX^k(Y) \rightarrow \lim_{\leftarrow} H_c^k(N_i) \rightarrow 0$$

where H_c stands for the cohomology with compact supports. Let $A \subset Y$, denote by $A_i = St(p_i(A), N_i)$. We may assume that $q_{i+1}^i(A_i) \subset A_{i+1}$. Then for relative coarse cohomology there is an exact sequence:

$$(*) \quad 0 \rightarrow \lim^1 H_c^{k-1}(N_i \setminus A_i) \rightarrow HX^k(Y, A) \rightarrow \lim_{\leftarrow} H_c^k(N_i \setminus A_i) \rightarrow 0.$$

The coarse cohomology and this exact sequence are defined for any coefficient group. Using coarse cohomology with integral coefficients one can define a *coarse cohomological dimension* [Dr1]

$$\text{X-dim } Y = \max\{k \mid HX^k(Y, A) \neq 0 \text{ for some } A \subset Y\} - 1.$$

The shift by one is needed to get the equality $X\text{-dim } \mathbf{R}^n = n$. We note that in [Dr1] this dimension was defined under a different name. Also in [Dr1] it was suggested to make a shift in the grading of the coarse cohomology in order to achieve the equality $X\text{-dim } \mathbf{R}^n = n$. Here we embed this shift in the definition of $X\text{-dim}$.

Proposition 4.1. *Let Y be a proper metric space, then*

$$\text{X-dim } Y \leq \text{asdim } Y.$$

Proof. Assume that $\text{asdim } Y \leq n$. There is an anti-Čech approximation of Y with polyhedra N_i of dimension $\leq n$. Then $H_c^k(N_i, A_i) = 0$ for any A_i for $k > n$. Therefore, $HX^k(Y, A) = 0$ for $k > n + 1$. Hence $X\text{-dim } Y \leq n$. \square

Proposition 4.2. *For every metric space Y ,*

$$\text{X-dim}(Y \times \mathbf{R}) = \text{X-dim } Y + 1.$$

Proof. Let $\text{X-dim } Y = n$ and let $HX^n(Y, A) \neq 0$. Since for all k ,

$$H_c^k(N_i \setminus A_i) = H_c^{k+1}((N_i \setminus A_i) \times \mathbf{R}),$$

we obtain that $HX^{n+1}(Y \times \mathbf{R}, A \times \mathbf{R}) \neq 0$. Thus, $\text{X-dim}(Y \times \mathbf{R}) \geq \text{X-dim } Y + 1$. In view of the inequality $\text{asdim}(Y \times \mathbf{R}) \leq \text{X-dim } Y + 1$, Proposition 4.1 implies the inequality $\text{X-dim}(Y \times \mathbf{R}) \leq \text{X-dim } Y + 1$. \square

We note that the coarse cohomological dimension can be defined with any (abelian) coefficient groups G . We will use the notation $\text{X-dim}_G Y$.

Let $\{G_i, \phi_i^{i+1}\}$ be an inverse system with bonding maps $\phi_i^{i+1} : G_{i+1} \rightarrow G_i$. For $k > i$ we denote by $\phi_i^k : G_k \rightarrow G_i$ the composition $\phi_i^{i+1} \circ \cdots \circ \phi_{k-1}^k$ and denote by

$$\phi_i^\infty : \lim_{\leftarrow} G_j \rightarrow G_i$$

the projection from the limit space to the i th factor.

Proposition 4.3. *Suppose that for an inverse sequence of countable groups*

$$\lim_{\leftarrow}^1 \{G_i, \phi_i^{i+1}\} \neq 0.$$

Then there is i_0 such that for every $i \geq i_0$ there is an element $\alpha_i \in G_i$ with $\phi_{i_0}^i(\alpha_i) \neq 0$ and $\alpha_i \notin \text{Im}(\phi_i^\infty)$.

Proof. Since $\lim^1 G_i$ is nonzero, the Mittag-Leffler condition does not hold for the system. Therefore there is i_0 such that the nested sequence

$$\text{Im}(\phi_{i_0}^{i_0+1}) \supset \text{Im}(\phi_{i_0}^{i_0+2}) \supset \text{Im}(\phi_{i_0}^{i_0+3}) \supset \dots$$

does not stabilize. In view of this we can take α_i such that $\phi_{i_0}^i(\alpha_i) \in \text{Im}(\phi_{i_0}^i) \setminus \text{Im}(\phi_{i_0}^k)$ for some k . \square

Proposition 4.4. *Suppose that a geodesic metric space Y that admits an anti-Čech approximation $\{N_i\}$ satisfies the equality $X - \dim_G Y = n$ and let $HX^{n+1}(Y, A; G) \neq 0$ for some A and some countable group G . Then the homomorphism*

$$\lim_{\leftarrow}^1 H_c^n(N_i \setminus A_i; G) \xrightarrow{\cong} HX^{n+1}(X, A; G)$$

in the short exact sequence (*) is an isomorphism.

Proof. We assume that all N_i are given the uniform geodesic metric.

Assume the contrary. Let $\alpha \in \lim_{\leftarrow} H_c^{n+1}(N_i \setminus A_i; G)$ and $\alpha \neq 0$. Let (α_i) be a thread representing α , $\alpha_i \in H_c^{n+1}(N_i \setminus A_i; G)$. From the definition of cohomology with compact supports it follows that there is a bounded open set $U_1 \subset N_1 \setminus A_1$ and an element $\gamma_1 \in H_c^{n+1}(U_1; G)$ which is taken by the inclusion homomorphism to α_1 . There is $r_1 > 0$ such that U_1 lies in the $r_1/2$ -neighborhood of A_1 , $U_1 \subset N_{r_1/2}(A_1)$. Let $W_1 = p_1^{-1}(U_1)$.

Since the projection $p_1 : X \rightarrow N_1$ is uniformly cobounded, the set $A^1 = p_1^{-1}(N_{r_1}(A_1))$ is in bounded distance to A . Then $HX^{n+1}(Y, A^1; G) \cong HX^{n+1}(Y, A; G)$. Moreover for large enough k there is an isomorphism of inverse sequences

$$\begin{array}{ccccccc} H_c^{n+1}(N_1 \setminus A_k; G) & \longleftarrow & H_c^{n+1}(N_1 \setminus A_{k+1}; G) & \longleftarrow & H_c^{n+1}(N_1 \setminus A_{k+2}; G) & \longleftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ H_c^{n+1}(N_1 \setminus A_k^1; G) & \longleftarrow & H_c^{n+1}(N_1 \setminus A_{k+1}^1; G) & \longleftarrow & H_c^{n+1}(N_1 \setminus A_{k+2}^1; G) & \longleftarrow & \dots \end{array}$$

There is a bounded open set $U_2 \subset N_k \setminus A_k^1$ and an element $\gamma_2 \in H_c^{n+1}(U_2; G)$ that goes to α_k under the inclusion homomorphism. Let $U_1^2 = (q_1^2)^{-1}(U_2)$ and $W_2 = p_1^{-1}(U_2^1)$. The commutative diagram

$$\begin{array}{ccc} H_c^{n+1}(N_1 \setminus A_1; G) & \longleftarrow & H_c^{n+1}(N_1 \setminus A_k; G) \\ \uparrow & & \uparrow \\ H_c^{n+1}(U_1^2; G) & \xleftarrow{(q_1^2|)^*} & H_c^{n+1}(U_2; G) \end{array}$$

implies that $(q_1^2|)^*(\gamma_2) \neq 0$. There is $r_2 > 0$ such that U_2 lies in $r_2/2$ -neighborhood of A_k and we continue in a similar fashion. As the result we construct a dispersed sequence of bounded open sets $W_i \subset Y$ and an anti-Čech approximation

$$(N_1, N_1 \setminus (\sqcup_j U_1^j)) \rightarrow (N_{k_2}, N_{k_2} \setminus (\sqcup_j U_2^j)) \rightarrow (N_{k_3}, N_{k_3} \setminus (\sqcup_j U_3^j)) \rightarrow \dots$$

of the pair $(Y, Y \setminus (\sqcup_i W_i))$ such that for all i there are nontrivial elements $\gamma_i \in H_c^{n+1}(U_i; G)$ that survive after translation to the first level. We may assume that these elements do not belong to the image of the projection from the higher level. This implies that the system is not Mittag-Leffler and hence $\lim^1 H_c^{n+1}(U_i; G) \neq 0$. Therefore, $HX^{n+2}(Y, Y \setminus (\sqcup_i W_i); G) \neq 0$ and hence $\text{X-dim}_G Y \geq n+1$. This contradicts to the assumption. \square

A metric space Y is *uniformly n -connected* if there is a control function $\rho : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, $\rho(t) \geq t$, such that for every $t > 0$ and every $y \in Y$ the inclusion $B_t(y) \rightarrow B_{\rho(t)}(y)$ induces zero homomorphism of k -dimensional homotopy groups for $k \leq n$. It is called *uniformly contractible* if there is a function $\rho : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, $\rho(t) \geq t$, such that for every $t > 0$ and every $y \in Y$ the inclusion $B_t(y) \rightarrow B_{\rho(t)}(y)$ is null-homotopic.

We recall that for every Y there is a natural through homomorphism $c : HX^k(Y; G) \rightarrow H_c^k(Y; G)$ [Ro1]:

$$HX^k(Y; G) \rightarrow \lim_{\leftarrow} H_c^k(N_i; G) \rightarrow H_c^k(Y; G).$$

Theorem 4.5 ([Ro1], page 33). *For a uniformly n -connected metric space Y the map $c : HX^k(Y; G) \rightarrow H_c^k(Y; G)$ is an isomorphism for $k \leq n$ and for any abelian coefficients group G .*

We denote the global cohomological dimension of a space Y with respect to the coefficient group G by

$$\text{gcd}_G Y = \max\{n \mid H_c^n(Y; G) \neq 0\}.$$

Proposition 4.6. *Let Y be a uniformly n -connected metric space of bounded geometry and let $\text{gld}_G Y \leq n$ for a countable group G . Then $\text{X-dim}_G Y \geq \text{gld}_G Y$.*

Proof. Assume the contrary: $\text{X-dim}_G Y = m < k = \text{gld}_G Y$. In view of Theorem 4.5 $HX^k(Y; G) = H_c^k(Y; G) \neq 0$ and hence, $m \geq k-1$. Then $m = k-1$. Since Y has bounded geometry, it has an anti-Čech approximation $\{N_i\}$. By Proposition 4.4 and Theorem 4.5 we obtain

$$\lim_{\leftarrow}^1 H_c^k(N_i; G) = HX^k(Y; G) = H_c^k(Y; G).$$

The last group is countable as the direct limit of countable groups. The first group is uncountable as a nonzero group which is a \lim^1 group of a sequence of countable groups [Ha]. We arrived to a contradiction. \square

Proposition 4.7. *Suppose that $\text{X-dim}_G Y = n$ for a proper geodesic metric space Y and a countable group G . Then there is a dispersed sequence $\{U_i\}$ of bounded open sets in Y such that $HX^{n+1}(\sqcup \bar{U}_i, \sqcup \partial U_i; G) \neq 0$. Moreover, there is an anti-Čech approximation $\{N_i, p_i\}$ of Y such that $U_i = p_i^{-1}(V_i)$, $V_i = \text{Int}(L_i)$, and $L_i \subset N_i$ is a finite subcomplex.*

Proof. Assume that $HX^{n+1}(Y, A; G) \neq 0$. Let

$$N_1 \xrightarrow{q_2^1} N_2 \xrightarrow{q_3^2} N_3 \xrightarrow{q_4^3} \dots N_i \xrightarrow{q_{i+1}^i} \dots$$

be a regular anti-Čech approximation of Y . By Proposition 4.4,

$$\lim_{\leftarrow}^1 H^n(N_i, A_i; G) \neq 0.$$

Note that $q_{i+1}^i : (N_i, A_i) \rightarrow (N_{i+1}, A_{i+1})$ is homotopic to $p_{i+1}^i : (N_i, A_i) \rightarrow (N_{i+1}, A_{i+1})$. Thus, $(q_{i+1}^i)^* = (p_{i+1}^i)^*$.

We apply Proposition 4.3 to obtain i_0 and $\alpha_i \in H^n(N_i, A_i; G)$. Without loss of generality we may assume that $i_0 = 1$. Then $(p_1^1)^*(\alpha_i) \neq 0$ and $\alpha_i \notin \text{Im}(p_\infty^i)^*$ where

$$(p_\infty^i)^* : \lim_{\leftarrow} H^n(N_k, A_k; G) \rightarrow H^n(N_i, A_i; G)$$

is the projection from the limit to i th factor in the inverse sequence

$$\{H^n(N_k, A_k; G), (p_{k+1}^k)^*\}.$$

Since $H^n(N_k, A_k; G) = H_c^n(N_k \setminus A_k; G)$ there is a open set $V_k \subset N_k \setminus A_k$ with compact closure \bar{V}_k and an element $\beta_k \in H_c^n(V_k; G)$ such that β_k goes to α_k under the inclusion homomorphism. We may assume that $V = \text{Int}L_k$ where L_k is a finite subcomplex of N_k . We define $U_i = p_i^{-1}(V_i)$. For $j \geq i$ we denote by $L_i^j = (p_j^i)^{-1}(L_j)$. For $j < i$ we denote $L_i^j = N(\mathcal{U}_j|_{\bar{U}_j})$, the nerve of the cover generating N_j restricted to \bar{U}_j . In view of $(*)$ it suffices to show that $\lim_{\leftarrow}^1 H^n(\sqcup_j(L_i^j, \partial L_i^j); G) \neq 0$. The inverse sequence $\{H^n(\sqcup_{j \geq 1}(L_i^j, \partial L_i^j); G)\}$ can be mapped epimorphically onto the sequence $\{H_c^n(\sqcup_{j \geq i}(L_i^j, \partial L_i^j); G)\}$. In view of the 6-term exact sequence for the inverse limit it suffices to show that

$$\lim_{\leftarrow}^1 \{H_c^n(\sqcup_{j \geq i}(L_i^j, \partial L_i^j); G)\} \neq 0.$$

Since for $j > i$ the element β_j goes to nonzero, the result follows. \square

Coarse cohomological dimension vs asymptotic. On subgroups of the reals \mathbf{R} we consider a natural norm $|\cdot|$. On the mod p group \mathbf{Z}_p we consider the zero semi-norm.

Theorem 4.8. *The following holds for every proper metric space Y with finite asymptotic dimension:*

- (1) $\text{asdim}_G Y \geq \text{X-dim}_G Y$ for every subgroup $G \subset \mathbf{Q}$;
- (2) $\text{asdim}_{\mathbf{Z}_p} Y = \text{X-dim}_{\mathbf{Z}_p} Y$ for all p .

Proof. (1) If $G = \mathbf{Z}$, the result follows from Theorem 3.6 and Proposition 4.1.

Now we assume that G is p -divisible for some p . Let $\text{X-dim}_G Y = n$ and let U_i be as in Proposition 4.5. Let

$$\sqcup_i(N_1^i, K_1^i) \xrightarrow{q_2^1} \sqcup_i(N_2^i, K_2^i) \xrightarrow{q_3^2} \sqcup_i(N_3^i, K_3^i) \rightarrow \dots$$

be an anti-Čech approximation of $\sqcup_i(\bar{U}_i, \partial U_i)$ where N_i are finite complexes. Then $\lim^1 H_c^n(\sqcup_i(N_j^i \setminus K_j^i); G) \neq 0$. Note that each group $H_c^n(\sqcup_i(N_j^i \setminus K_j^i); G) = \bigoplus_i H^n(N_j^i, K_j^i; G)$ is countable. Hence the system is not Mittag-Lefler. Without loss of generality, we may assume that for any m there are $i \geq m$ and $k = k(i)$ and an element $\gamma_i \in H^n(N_{k(i)}^i, K_{k(i)}^i; G)$ such that $(q_{k(i)}^1)^*(\gamma_i) \neq 0$ and $\gamma_i \notin \text{Im}(q_{k(i)+1}^{k(i)})$. By dividing γ_i by some power of p we may achieve that γ_i is represented by a cocycle with the norm ≤ 1 . This defines a nontrivial element of $AH_b^n(\sqcup_i(N_{k(i)}^i, K_{k(i)}^i); G)$ and of $AH^n(\sqcup \bar{U}_i, \sqcup \partial U_i; G)$. Thus, $\text{asdim}_G Y \geq n$.

(2) The same argument works to show that $\text{asdim}_{\mathbf{Z}_p} Y \geq \text{X-dim}_{\mathbf{Z}_p} Y$.

Let $\text{asdim}_{\mathbf{Z}_p} Y = n$. We apply Lemma 3.7 to obtain a dispersed family of open set $\{U_i\}$ with $AH^n(\sqcup \bar{U}_i, \sqcup \partial U_i; \mathbf{Z}_p) \neq 0$. Without loss of generality we may assume that $U_i = p_i^{-1}(V_i)$ where $\{N_i, p_i\}$ is an anti-Čech approximation and $\bar{V}_i \subset N_i$ are subcomplexes. Furthermore, there is i_0 and

$$\alpha \in H_b^n(\sqcup_{i \geq i_0}(\bar{V}_i, \partial V_i); \mathbf{Z}_p) = H^n(\sqcup_{i \geq i_0}(\bar{V}_i, \partial V_i); \mathbf{Z}_p) = \prod_{i \geq i_0} H^n(\bar{V}_i, \partial V_i; \mathbf{Z}_p)$$

such that α defines a nonzero element of $AH^n(\sqcup \bar{U}_i, \sqcup \partial U_i; \mathbf{Z}_p) \neq 0$. If $\alpha = (\alpha_i)_{i \geq i_0}$, this implies that for every function $\kappa : \mathbf{N} \rightarrow \mathbf{N}$, $\kappa(i) > i$, the sequence $((p_i^{\kappa(i)})^*(\alpha_i))_{i \geq i_0}$ is not eventually zero. We show that there is k such that for $i > k$ the image $(p_i^k)^*(\alpha_i)$ is nonzero for infinitely many i . Let $J_k = \{i \in \mathbf{N} \mid (p_i^k)^*(\alpha_i) \neq 0\}$. Note that $J_k \subset J_{k+1}$ and $i \in J_i$. If each J_k is finite we can define $\kappa : \mathbf{N}_{\geq i_0} \rightarrow N_{\geq i_0}$ by the formula $\kappa(J_k \setminus J_{k-1}) = k - 1$. Then $\kappa \rightarrow \infty$ and the sequence $((p_i^{\kappa(i)})^*(\alpha_i))_{i \geq i_0}$ is zero which contradicts to the assumption.

Then it follows that $\lim^1 H_c^n(\sqcup_{j > i}(p_i^j)^{-1}(V_i); \mathbf{Z}_p) \neq 0$. Therefore $HX^{n+1}(Y, Y \setminus \sqcup U_i; \mathbf{Z}_p) \neq 0$ and hence $\text{X-dim}_{\mathbf{Z}_p} Y \geq n$. Thus, $\text{asdim}_{\mathbf{Z}_p} Y \leq \text{X-dim}_{\mathbf{Z}_p} Y$. \square

Applications to dimension of discrete groups.

Theorem 4.9. *Let Γ be a discrete group with finite $B\Gamma$, then $\text{X-dim } \Gamma = cd(\Gamma)$.*

Proof. First we show that $\text{X-dim } \Gamma \geq cd(\Gamma)$. Let $cd(\Gamma) = n$. Since $cd(\Gamma) = gld_{\mathbf{Z}}\Gamma$ [Br] and $E\Gamma$ is uniformly contractible, by Proposition 4.6 we obtain $\text{X-dim}_{\mathbf{Z}} E\Gamma = \text{X-dim}_{\mathbf{Z}} \Gamma \geq n$.

Let $\text{X-dim } \Gamma = n$ and let $cd\Gamma < n$. By crossing Γ with \mathbf{Z} and applying Proposition 4.2 we may assume that $n \geq 3$. Then we have $\dim E\Gamma < n$. Let $\{N_i, p_{i+1}^i, q_{i+1}^i\}$ be a regular anti-Čech approximation of $E\Gamma$. By Proposition 4.7 there is a dispersed sequence of open sets $U_i = p_i^{-1}V_i$ and elements $\alpha_i \in H_c^n(V_i)$ with $(p_i^1)^*(\alpha_i) \neq 0$. Let W_i be the regular neighborhood of \bar{V}_i in N_i , i.e., W_i is the star neighborhood of \bar{V}_i in the second barycentric subdivision of N_i . Let A_i be the regular neighborhood of ∂V_i . Since the pair (W_i, A_i) is homotopy equivalent to $(\bar{V}_i, \partial V_i)$, the element α_i lives in $H^n(W_i, A_i)$. Since $E\Gamma$ is uniformly contractible there are lifts $s_i : V_i^1 = (p_i^1)^{-1}(\bar{V}_i) \rightarrow E\Gamma_i$ such that $p_1 s_i$ is r -closed to the identity where r is the same for all i . Since the Lipschitz constant of p_i^1 tends to zero, for large enough i the maps $p_i^1 : (\bar{V}_i^1, \partial V_i^1) \rightarrow (W_i, A_i)$ and $p_i s_i : (V_i^1, \partial V_i^1) \rightarrow (W_i, A_i)$ are homotopic. Hence $(p_i^1)^*(\alpha_i) = s_i^* p_i^*(\alpha_i)$. Since $\dim E\Gamma < n$, s_i^* is zero homomorphism and hence $s_i^* p_i^*(\alpha_i) = 0$. We arrived to a contradiction. \square

We recall that in the group theoretic language the groups with finite $B\Gamma$ are called the groups of the type *FL*. Also we recall that a finitely presented group is called of the type *FP* if $B\Gamma$ is dominated by a finite complex.

Proposition 4.10. *Let Γ be a discrete group of the type *FP*. Then $\text{asdim } \Gamma \geq cd(\Gamma)$.*

Proof. We may assume that $\text{asdim } \Gamma < \infty$. Let $cd(\Gamma) = n$. Then $H^n(\Gamma; \mathbf{Z}\Gamma) \neq 0$ [Br]. The condition $\Gamma \in \text{FP}$ is equivalent to the existence of a homotopy domination $r : K \rightarrow B\Gamma$ by a finite complex. We may assume that r induces an isomorphism of the fundamental groups and $\pi_i(K) = 0$ for $1 < i \leq n$. Then $H^n(\Gamma; \mathbf{Z}\Gamma) = H_c^n(\tilde{K})$ where \tilde{K} is the universal cover of K . Since the space \tilde{K} is uniformly n -connected, by Proposition 4.6 we obtain $\text{X-dim}_{\mathbf{Z}} \tilde{K} \geq gld_{\mathbf{Z}} \tilde{K} \geq n$. Since \tilde{K} coarsely equivalent to $\pi_1(K) = \Gamma$, we obtain by Theorem 3.6 and Proposition 4.8(1) that $\text{asdim } \Gamma \geq \text{asdim}_{\mathbf{Z}} \Gamma = \text{asdim}_{\mathbf{Z}} \tilde{K} \geq \text{X-dim}_{\mathbf{Z}} \tilde{K} \geq n$. \square

REMARK. The same argument works for the groups Γ with finite $cd(\Gamma)$ and $B\Gamma$ having finite skeleton in each dimension.

We recall that a group Γ is of the type *VFP* if it admits a subgroup of finite index of the type *FP* [Br].

Corollary 4.11. *$vc\Gamma \leq \text{asdim } \Gamma$ for groups Γ of the type *VFP*.*

Famous Stollings-Swan theorem implies the following.

Corollary 4.12. *Every group Γ of type VFP with $\text{asdim } \Gamma \leq 1$ is virtually free.*

This result without the VFP restriction was proven independently by Januszkiewicz and Swiatkowski [JS] and Gentimis [G].

Theorem 4.13. *Suppose that $\text{asdim}_{\mathbf{Z}} \Gamma = \text{asdim}_{\mathbf{Z}_p} \Gamma$ for a group Γ of FL type with finite asymptotic dimension and some prime p . Then $\text{asdim } \Gamma = \text{cd} \Gamma$.*

Proof. By Theorem 4.8(2) and Theorem 3.6, $\text{X-dim}_{\mathbf{Z}_p} \Gamma = \text{asdim}_{\mathbf{Z}_p} \Gamma = \text{asdim } \Gamma$. By Theorem 4.9 $\text{cd} \Gamma = \text{X-dim}_{\mathbf{Z}} \Gamma \geq \text{X-dim}_{\mathbf{Z}_p} \Gamma = \text{asdim } \Gamma$. By Proposition 4.10 $\text{asdim } \Gamma = \text{cd} \Gamma$. \square

§5 A COUNTEREXAMPLE TO THE ASYMPTOTIC ANALOG OF MORITA'S THEOREM

The main idea of the construction can be demonstrated on the following example. Let p, q be two mutually prime numbers and let D be a 2-disc with two disjoint discs removed from its interior. Let S_1^1 and S_2^1 be the boundaries of the removed discs and let S^1 be the external boundary. We consider a free \mathbf{Z}_p action on S_1^1 and a free \mathbf{Z}_q action on S_2^1 . Let M denote the quotient space which is obtained from D by factorization of S_1^1 and S_2^1 to the orbit spaces. Since the equation $mp + nq = 1$ has a solution in integers, there is a retraction of M to S^1 . The degree of any such retraction restricted to the circle S_1^1/\mathbf{Z}_p is m and the degree of the restriction to S_2^1/\mathbf{Z}_q is n for some m, n satisfying $mp + nq = 1$. Thus, for large p and q these degrees have to be also large. This allows to construct a uniform complex M that admits a retraction to the 'boundary' and such that every such retraction has the Lipschitz constant large.

Now we present the construction. Fix two primes $p > q^2$. For every natural k we define a 2-dimensional complex M_k as follows. Let T_p be the mapping cylinder of the degree p map $z^p : S^1 \rightarrow S^1$. For a subdivisions of S^1 into p^{i+1} and p^i pieces we fix a mapping cylinder triangulation T_p^i on T_p . Fixing orientation on S^1 we may assume that the complex T_p^i is oriented. We consider the union $M_k^p = T_p^{k-1} \cup T_p^{k-2} \cup \dots \cup T_p^1$ with identification of the image of T_p^i with the domain of T_p^{i-1} . Let $s \in S^1$ be a base point. We may assume that it is taken by the map z^p to the base point (consider $s = 1$). Let $M'_k = M_k^p \vee M_k^q$ be the wedge of these complexes with the base vertices located in the domains. Let M be a mapping cylinder of the map $\phi : S^1 \rightarrow S^1 \vee S^1$ that collapses two points in S^1 . We may supply M with a triangulation having the following properties: (1) every vertex belongs to at most p edges; (2) the domain $S^1 \subset M$ of ϕ has 3 edges; (3) one circle in the target space $S^1 \vee S^1 \subset M$ of ϕ has $(3p)^k$ edges and the other has $(3q)^k$ edges. Glue M along the target to M'_k to obtain \tilde{M}_k . Then we consider a 2-simplex Δ subdivided in four 2-simplices by middle points of the edges and delete the interior of

the middle simplex $D = \Delta \setminus \text{Int}\sigma$. Glue \tilde{M}_k to $\partial\sigma$ along the domain of M to obtain M_k . We denote $\partial M_k = \partial\Delta \subset D$ and will refer to this set as to the boundary of M_k . The end circles of M_p^k and M_q^k we call the boundaries of p -hole and q -hole in M_k respectively and denote them by S_p^1, S_q^1 . Note that they are triangles, i.e., they have three edges. Assume that M_k is given a metric of uniform simplicial complex. We fix a map $\phi_k : M_k \rightarrow \Delta$ which is simplicial with respect to the midpoint subdivision of Δ and such that the original vertices of Δ have exactly one preimage each. Clearly, this map is 1/2-Lipschitz for the uniform metric on M_k .

We define complexes $M_{k,k-1,\dots,k-i}$ for $i = 0, \dots, k-1$ by induction on i . Assume that the simplicial complex $M_{k,k-1,\dots,k-i}$ is already constructed. For every 2-simplex $\Delta \subset M_{k,k-1,\dots,k-i}$ we delete its interior and glue instead a copy of $M_{k,k-1,\dots,k-i-1}$ along the boundary $\partial M_{k,k-1,\dots,k-i-1}$. This defines a map

$$\phi_{k,\dots,k-i-1} : M_{k,\dots,k-i-1} \rightarrow M_{k,\dots,k-i}$$

which is 1/2-Lipschitz. For $j > i$ we denote $\phi_j^i = \phi_{k,\dots,j} \circ \phi_{k,\dots,j-1} \circ \dots \circ \phi_{k,i}$.

Proposition 5.1. *The following holds true:*

- (1) *For every k there is a retraction $r : M_k \rightarrow \partial M_k$.*
- (2) *The Lipschitz constant of every retraction r is greater than q^k .*
- (3) *The Lipschitz constant of every map $f : M_k \rightarrow S^1$ having the restriction $f|_{\partial M_k}$ with the nonzero degree $\deg(f|_{\partial M_k}) \neq 0$ is greater than q^k . Moreover, $\text{Lip}(f|_{\partial M_k \cup S_p^1 \cup S_q^1}) \geq q^k$.*
- (4) *The Lipschitz constant of every map $f : M_{k,\dots,i} \rightarrow S^1$ with the nonzero degree restriction $\deg(f|_{\partial M_{k,\dots,i}}) \neq 0$ is greater than q^i .*

Here S^1 is given a metric of the boundary of the standard 2-simplex.

Proof. (1). Note that M_k is homotopy equivalent to a 2-complex obtained from the wedge $S_e^1 \vee S_p^1 \vee S_q^1$ by attaching a 2-cell along the loop $\bar{e}a^{p^k}b^{q^k}$. Let n and m be such natural numbers that $np^k + mq^k = 1$. We consider maps $r_p : S_p^1 \rightarrow S^1$ and $r_q : S_q^1 \rightarrow S^1$ of degree n and m respectively. Then the map $\text{id}_{S^1} \cup r_p \cup r_q : S_e^1 \vee S_p^1 \vee S_q^1 \rightarrow S^1$ has an extension $r : M_k \rightarrow \partial M_k$ since the attaching map composed with it has the degree $0 = -1 + np^k + mq^k$.

- (2). Note that $\text{Lip}(r) \geq \text{Lip}(r_q) \geq |m| \geq (p/q)^k > q^k$.
- (3). Let $d = \deg(f|_{\partial M_k})$ and let $d_p = \deg(f|_{S_p^1})$, and $d_q = \deg(f|_{S_q^1})$. Then $d_p p^k + d_q q^k = d$. If one of the coefficients d_p, d_q is zero, then $\text{Lip}(f) \geq \text{Lip}(f|_{\partial M_k}) \geq d \geq q^k$. If both are nonzero, then they are divisible by d and $d_q/d \geq q^k$ as above. Hence, $\text{Lip}(f) \geq q^k$.

(4). By induction on $k - i$. For $i = k$, the result is proven in (2). For $i < k$ we consider two cases. First we consider the case when the restriction of f to one of the end circles in $M_{k,\dots,i}$ has nonzero degree. Then the restriction of f to a copy of M_i satisfies the conditions of (3). Therefore, $\text{Lip}(f|_{M_i}) \geq q^i$.

Next we assume that f has zero degree on every end circle of $M_{k,\dots,1}$. Then, f defines a map $\tilde{f} : M_{k,\dots,i+1} \rightarrow \partial\Delta^2$ which agrees with f on the 1-dimensional skeleton of $M_{k,\dots,i+1}^{(1)} \subset M_{k,\dots,i}$. By the induction assumption $\text{Lip}(\tilde{f}) \geq q^{i+1}$. Since $\text{Lip}(\tilde{f}) = \text{Lip}(\tilde{f}|_{M_{k,\dots,i+1}^{(1)}})$, we obtain $\text{Lip}(f) \geq q^{i+1}$. \square

Let $q_k : M_k \rightarrow \Delta$ be a simplicial approximation of $\phi_k : M_k \rightarrow \Delta$. Let $n_k \in \mathbf{N}$, denote by $\xi_k : [0, n_k] \rightarrow [0, 1]$ the continuous map that collapses $[0, n_k - 1]$ to 0 and maps $[n_k - 1, n_k]$ isometrically onto $[0, 1]$. Note that ξ_k is a simplicial approximation of an orientation preserving homeomorphism $[0, n_k] \rightarrow [0, 1]$ where $[0, n_k]$ is subdivided into the intervals of length one. Let $g_k = q_k \times \xi_k : M_k \times [0, n_k] \rightarrow \Delta \times [0, 1]$.

Proposition 5.2. *There is $\lambda > 0$ such that for every k there is $n_k \in \mathbf{N}$ and a λ -Lipschitz map $f_k : M_k \times [0, n_k] \rightarrow S^2$ such that $f_k|_{\partial(M_k \times [0, n_k])} = g_k|_{\partial(M_k \times [0, n_k])}$ where $\partial(M_k \times [0, n_k]) = \partial M_k \times [0, n_k] \cup M_k \times \{0, n_k\}$, $S^2 = \partial(\Delta^2 \times [0, 1])$ with the l_1 -product metric.*

Proof. Let $m_k = \min\{\|\gamma\| \mid \delta\gamma = q_k^*(1_\Delta)\}$ where $1_\Delta \in C^2(\Delta)$ is a simplicial cocycle that takes 1 on Δ . According to Proposition 5.1(1) $m_k < \infty$. Assume that this m_k is attained on a cochain γ_k . We take $n_k = m_k!$ and consider the extension problem

$$\begin{array}{ccc} \partial(M_k \times [0, n_k]) & \xrightarrow{|g_k|} & \partial(\Delta \times [0, 1]) \\ \subset \downarrow & & \\ M_k \times [0, n_k]. & & \end{array}$$

We note that the product $M_k \times [0, n_k]$ has a natural structure of a 3-dimensional cell complex where $[0, n_k]$ is subdivided into unit intervals, and the map g_k takes its 2-skeleton to $\partial(\Delta \times [0, 1])$. Then the obstruction cocycle equals $c = g_k^*(1_{\Delta \times [0, 1]})$ where $1_{\Delta \times [0, 1]}$ is the cellular 3-cocycle on $\Delta \times [0, 1]$ that takes this 3-cell to one. We construct a 2-cochain $\beta_k \in C^2(M_k \times [0, n_k])$ with $\delta\beta_k = c$ and with $|\beta_k| \leq 4$.

For every 1-simplex e in M_k with $\gamma_k(e) \neq 0$ we set $\beta_k(e \times [im, im + 1]) = \text{sgn}(\gamma_k(e))$ for the integer $m = n_k/|\gamma_k(e)|$ and $i = 0, \dots, |\gamma_k(e)| - 1$, and set $\beta_k(e \times [s, s + 1]) = 0$

for all other cells. Here $\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$

Note that $\beta_k(e \times [0, n_k]) = \gamma_k(e)$. Indeed,

$$\beta_k(e \times [0, n_k]) = \sum_i \beta_k(e \times [im, im + 1]) = |\gamma_k(e)| sgn(\gamma_k(e)) = \gamma_k(e).$$

For every 2-simplex $\sigma \subset M_i$ we set $\beta_k(\sigma \times \{l\}) = -\beta_k(\partial\sigma \times [0, l])$ for all natural $l < n_k$ and we define $\beta_k(\sigma \times \{0\}) = \beta_k(\sigma \times \{n_k\}) = 0$.

Show that $\delta\beta_k = c = g_k^*(1_{\Delta \times [0,1]})$. Indeed,

$$\delta\beta_k(\sigma \times [s-1, s]) = \beta_k(\partial\sigma \times [s-1, s] + \sigma \times \{s\} - \sigma \times \{s-1\}) = \beta_k(\partial\sigma \times [s-1, s]) -$$

$$\beta_k(\partial\sigma \times [0, s]) + \beta_k(\partial\sigma \times [0, s-1]) = 0 = g_k^*(1_{\Delta \times [0,1]})(\sigma \times [s-1, s])$$

for $s < n_k$. For $s = n_k$ we obtain

$$\delta\beta_k(\sigma \times [s-1, s]) = \beta_k(\partial\sigma \times [s-1, s]) + \beta_k(\partial\sigma \times [0, s-1]) = \beta_k(\partial\sigma \times [0, n_k]) = \gamma_k(\partial\sigma) =$$

$$\delta\gamma_k(\sigma) = q_k^*(1_{\Delta})(\sigma) = (q_k \times \xi)^*(1_{\Delta \times [0,1]})(\sigma \times [s-1, s]).$$

We show that the cochain β_k is bounded. Let $\partial\sigma = a + b + c$ where a, b, c are sides (with signs) of a 2-simplex $\sigma \subset M_k$. Note that

$$|\beta_k(\sigma \times \{l\})| = |\beta_k(\partial\sigma \times [0, l])| = |\beta_k(a \times [0, l]) + \beta_k(b \times [0, l]) + \beta_k(c \times [0, l])| =$$

$$|sgn(\gamma_k(a))[\frac{l|\gamma_k(a)|}{n_k}] + sign(\gamma_k(b))[\frac{l|\gamma_k(b)|}{n_k}] + sgn(\gamma_k(c))[\frac{l|\gamma_k(c)|}{n_k}]| \leq$$

$$\frac{l}{n_k}(\gamma_k(a) + \gamma_k(b) + \gamma_k(c)) + 3 \leq \frac{l}{n_k}(q_k^*(1_{\Delta})(\sigma)) + 3 \leq 4$$

where $[x]$ denote the integral part of x .

By the Obstruction Theory (see Proposition 3.4) the map g_k restricted to the 2-skeleton of $M_k \times [0, n_k]$ can be changed on 2-cells lying in $M_k \times [0, n_k] \setminus \partial(M_k \times [0, n_k])$ without changing on the 1-skeleton by means of the cochain β_k in such a way that a new map is $\mu(|\beta_k|)$ -Lipschitz and it is extendible over $M_k \times [0, n_k]$ to a λ -Lipschitz map $f_k : M_k \times [0, n_k] \rightarrow S^2$. \square

Theorem 5.3. *There is a proper geodesic metric space Y of bounded geometry with dimensions $\text{asdim } Y = 2$ and $\text{asdim}(Y \times \mathbf{R}) = 2$.*

Proof. Let Y_0 be the disjoint union of the uniform complexes $M_{k,\dots,1}$. To make it into a geodesic metric space we attach $M_{k,\dots,1}$ to the half-line \mathbf{R}_+ at the point $2^k \in \mathbf{R}_+$ for all k :

$$Y = \mathbf{R}_+ \cup_k M_{k,\dots,1}.$$

In view of Theorem 3.6 it suffices to prove that $\text{asdim } Y \geq 2$. We show that

$$AH^2(\sqcup_k M_{k,\dots,1}, \sqcup_k \partial M_{k,\dots,1}) \neq 0.$$

We note that for every k the projection ϕ_k^i takes $M_{k,\dots,i}$ to Δ^2 in such a way that it is a homeomorphism on the boundary $\partial M_{k,\dots,i} \cong \partial \Delta^2$. Let $i : \mathbf{N}_+ \rightarrow \mathbf{N}_+$, $i(k) \leq k$, $\lim i(k) = \infty$ and let $q_i : \sqcup_k M_{k,\dots,i(k)} \rightarrow \sqcup_k \Delta^2$ be a simplicial approximation of $\sqcup_k \phi^i(k)$. We show that for every $i : \mathbf{N}_+ \rightarrow \mathbf{N}_+$, $i(k) \leq k$, $\lim i(k) = \infty$,

$$q_i^* : AH_b^2(\sqcup_k \Delta^2, \sqcup_k \partial \Delta^2) \rightarrow AH_b^2(\sqcup_k M_{k,\dots,i(k)}, \sqcup_k \partial M_{k,\dots,i(k)})$$

takes the fundamental class μ to nonzero element. Then

$$AH_b^2(\sqcup_k \Delta^2, \sqcup_k \partial \Delta^2) \rightarrow AH_b^2(\sqcup_k M_{k,\dots,1}, \sqcup_k \partial M_{k,\dots,1})$$

will be a nonzero homomorphism.

Indeed, the image $q_i^*(1_{\Delta^2})$ of the fundamental cocycle is the obstruction cocycle C_f for a retraction $f : M_{k,\dots,i(k)} \rightarrow \partial \Delta^2$ defined on the 1-skeleton as q_i . By Proposition 5.1(1) and the Obstruction Theory, $C_f = \delta \gamma_k$ for some simplicial cochain $\gamma_k \in C^1(M_{k,\dots,i(k)})$. By Proposition 5.1 (4) and the Obstruction Theory, $\|\gamma_k\| \geq q^i(k) - 1$. Since $i(k) \rightarrow \infty$, the cocycle $q^*(1_{\Delta^2})$ defines a nontrivial element in $AH_b^2(\sqcup_k M_{k,\dots,i(k)}, \sqcup_k \partial M_{k,\dots,i(k)})$.

Now we show that $\text{asdim}(Y_0 \times \mathbf{R}) \leq 2$. We construct an anti-Čech approximation of $Y_0 \times \mathbf{R}$ by 2-dimensional complexes. We recall that the map $\phi_j^i : M_{k,\dots,j} \rightarrow M_{k,\dots,i}$ is defined for $k \geq j \geq i$. Let $Y_i = M_i \sqcup M_{i+1,i} \sqcup M_{i+2,i+1,i} \sqcup \dots$. We denote

$$\psi_i^1 = \sqcup_{k \geq i} \phi_i^1 : \sqcup_{k \geq i} M_{k,\dots,1} = Y_0 \rightarrow Y_i$$

and

$$\psi_i = \phi_i \sqcup \phi_{i+1,i} \sqcup \phi_{i+2,i+1,i} \sqcup \dots : Y_i \rightarrow \Delta \sqcup Y_{i+1}.$$

We consider the map $\psi_i \times \frac{1}{n_i} : Y_i \times \mathbf{R} \rightarrow (\Delta \sqcup Y_{i+1}) \times \mathbf{R}$ where n_i is from Proposition 5.2. We subdivide \mathbf{R} in the first product into the interval of the form $[ln_i, l(l+1)n_i]$ and subdivide \mathbf{R} in the second product into the unit intervals to turn $\psi_i \times \frac{1}{n_i}$ into a cellular map. By Proposition 5.2 there is a λ -Lipschitz sweeping $\xi_i : Y_i \times \mathbf{R} \rightarrow ((\Delta \sqcup Y_{i+1}) \times \mathbf{R})^{(2)} = N_i$ onto the 2-skeleton. Then the composition $\xi_i \circ (\psi_i^1 \times \frac{1}{i}) : Y_0 \times \mathbf{R} \rightarrow N_i$ is a uniformly cobounded λ/i -Lipschitz map onto a 2-dimensional complex. To make it into a genuine anti-Čech approximation one needs to triangulate all the prisms in N_i . \square

Theorem 5.3 together with Proposition 4.2 implies

Corollary 5.4. *If Y as above, then $X - \dim Y = 1$.*

§6 A CONSTRUCTION OF LARGE SIMPLICIAL COMPLEXES

We present here a construction of infinite locally finite uniform simplicial complexes. Let $\{\phi_k : M_k \rightarrow \Delta^n, \chi_k : M_k \rightarrow \Delta^n\}$ be a sequence of maps of n -dimensional simplicial complexes such that for each k , χ_k is a light simplicial map (and hence a retraction) and ϕ_k is a simplicial map to a some subdivision τ_k of the of Δ^n such that $\text{mesh}(\tau_k) < \delta < 1$ for fixed δ for all k . We recall that a map ϕ is called *light* if all point preimages are 0-dimensional. A light simplicial map $\phi : M \rightarrow \Delta^n$ is an isomorphism on every n -simplex $\sigma \subset M$. Williams calls the complexes that admit a light simplicial map onto the simplex Δ^n as *complexes over Δ^n* [Wi].

Let $s_k : \Delta^n \rightarrow M_k$ be sections of χ_k .

Consider the following diagram:

$$\begin{array}{ccccccccc}
M_1 & \xleftarrow{\quad \rightarrow \quad} & M_{2,1} & \xleftarrow{\quad \rightarrow \quad} & M_{3,2,1} & \xleftarrow{\quad \rightarrow \quad} & \dots M_{k,\dots,1} & \xleftarrow{\quad \rightarrow \quad} & M_{k+1,\dots,1} \xleftarrow{\quad \rightarrow \quad} \dots \\
\downarrow \phi_1 & & \downarrow \phi_{2,1} & & \downarrow \phi_{3,2,1} & & \phi_{k,\dots,1} \downarrow & & \phi_{k+1,\dots,1} \downarrow \\
\Delta^n & \xleftarrow[\chi_2]{\quad \rightarrow \quad} & M_2 & \xleftarrow{\quad \rightarrow \quad} & M_{3,2} & \xleftarrow{\quad \rightarrow \quad} & \dots M_{k,\dots,2} & \xleftarrow{\quad \rightarrow \quad} & M_{k+1,\dots,2} \xleftarrow{\quad \rightarrow \quad} \dots \\
& & \phi_2 \downarrow & & \phi_{3,2} \downarrow & & \phi_{k,\dots,2} \downarrow & & \phi_{k+1,\dots,2} \downarrow \\
& & \Delta^n & \xleftarrow[\chi_3]{\quad \rightarrow \quad} & M_3 & \xleftarrow{\quad \rightarrow \quad} & \dots M_{k,\dots,3} & \xleftarrow{\quad \rightarrow \quad} & M_{k+1,\dots,3} \xleftarrow{\quad \rightarrow \quad} \dots \\
& & & & & & \ddots & & \\
& & & & & & \phi_k \downarrow & & \phi_{k,k-1} \downarrow \\
& & & & & & \Delta^n & \xleftarrow[\chi_{k+1}]{\quad \rightarrow \quad} & M_{k+1} \xleftarrow{\quad \rightarrow \quad} \dots \\
& & & & & & \phi_{k+1} \downarrow & & \\
& & & & & & \Delta^n & \xleftarrow{\quad \rightarrow \quad} & \dots
\end{array}$$

where all horizontal left arrows are generated by the maps χ_k by taking pull-back, all horizontal right arrows are generated by sections $s_k : \Delta^n \rightarrow M_k$ of maps χ_k , and all squares are pull-back diagrams. We define simplicial complexes Y_k as the direct limit of embeddings:

$$Y_k = \lim_{\rightarrow} \{M_k \rightarrow M_{k+1,k} \rightarrow M_{k+2,k+1,k} \rightarrow \dots\}.$$

Note that for $k < l$ there are the natural projections $\phi_l^k : Y_k \rightarrow Y_l$ obtained as the direct limits in the above diagram. We denote the top horizontal sections by $\sigma_k : M_{k,\dots,1} \rightarrow M_{k+1,\dots,1}$. Denote by

$$Y = Y_1 = \lim_{\rightarrow} \{ M_1 \xrightarrow{\sigma_1} M_{2,1} \xrightarrow{\sigma_2} M_{3,2,1} \rightarrow \dots \}.$$

We fix a simplicial approximation $\rho_k : \tau_k \rightarrow \Delta^n$ of the identity map $id : |\tau_k| \rightarrow \Delta$. It defines a simplicial approximation q_k^{k-1} of the map ϕ_k^{k-1} .

We assume that all Y_k are given the uniform geodesic metrics.

Proposition 6.1. *The family*

$$\{\phi_k^1 : Y \rightarrow Y_k, q_{k+1}^k, \phi_{k+1}^k\}$$

is a regular anti-Čech approximation of Y .

Proof. We consider $\mathcal{U}_k = (\phi_k^1)^{-1}\{Ost(v, Y_k)\}$. By the construction ϕ_k^1 is a simplicial map to a $k-1$ -iterated δ -subdivision of a uniform complex. Therefore, ϕ_k^1 is δ^{k-1} -Lipschitz on every simplex of Y . Since the metric on Y is geodesic, it is δ^{k-1} -Lipschitz. Since $\delta^k \rightarrow 0$, the Lebesgue numbers $L(\mathcal{U}_k)$ tends to infinity. Since ϕ_{k+1}^k is simplicial with respect to a subdivision, for every vertex $v \in Y_k$ there is a vertex $u \in Y_{k+1}$ such that

$$\phi_{k+1}^k(Ost(v, Y_k)) \subset Ost(\phi_{k+1}^k(v), \tau_{k+1}) \subset Ost(u, Y_{k+1}).$$

Therefore, $Ost(v, Y_k) \subset (\phi_{k+1}^k)^{-1}(Ost(u, Y_{k+1}))$ and hence

$$(\phi_k^1)^{-1}(Ost(v, Y_k)) \subset (\phi_{k+1}^k)^{-1}(Ost(u, Y_{k+1}))$$

and the condition $\mathcal{U}_k \prec \mathcal{U}_{k+1}$ is checked. \square

REMARK. The space Y has natural compactification

$$\bar{Y} = \lim_{\leftarrow} \{M_1 \leftarrow M_{2,1} \leftarrow M_{3,2,1} \leftarrow \dots\}.$$

The complexes M_k constructed in §5 can be considered to be oriented. Let $\chi : M_k \rightarrow \Delta^2$ be a simplicial map defined by the orientation. We consider the following triangulation τ of the 2-simplex Δ^2 : First, we take the midpoint subdivision of Δ^2 and then take the cone subdivision of the central 2-simplex. Let $\phi_k : (M_k, S^1) \rightarrow (\Delta^2, \partial\Delta^2)$ be a simplicial map to the subdivision τ of Δ^2 which takes M'_k to the center and maps D to D by the identity map. Then Theorem 5.3 can be stated as follows.

Theorem 6.2. *The family $\{\phi_k : M_k \rightarrow \Delta^2, \chi_k : M_k \rightarrow \Delta^2\}$ for any choice of sections $s_k : \Delta^2 \rightarrow M_k$ defines the space Y as above with an anti-Čech approximation $\{\phi_k^1 : Y \rightarrow Y_k\}$ such that $\text{asdim}(Y \times \mathbf{R}) = \text{asdim } Y = 2$.*

Note that if $s_k(\Delta^2) \cap \partial M_k = \emptyset$ for all k then the boundary of the complex Y is homeomorphic to the circle S^1 .

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